2020 B
Week 5 (Feb. 9)

- Triple Integral

the development of triple integral is parallel to that of double integral. We will be brief.

A rectangular box

$$B = [a, b] \times [c, d] \times [e, f].$$

A partition P is

$$a = x_0 < x_1 < \cdots < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_m = d$$
$$e = z_0 < z_1 < \cdots < z_\ell = f$$

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

$$|B_{ijk}| = \Delta x_i \Delta y_j \Delta z_k.$$

For a bounded function f in B, its Riemann sum is

$$R(f, P) = \sum_{i,j,k} f(p_{ijk}) |B_{ijk}|, \quad \text{where}$$

$p_{ijk} \in B_{ijk}$ is a tag point.

f is called <u>integrable</u> if there is a real number I so that whenever $\|P\| \to 0$,

$$R(f, P) \to I.$$

Here $\|P\| = \max \{\Delta x_i, \Delta y_j, \Delta z_k\}$.

Notation: $I = \iiint_B f, \quad \iiint_B f\, dV, \quad \iiint_B f(x,y,z)\, dV(x,y,z).$

$\iiint_B f$ is the integral of f over B.

- All continuous functions in B are integrable.
- More generally, all piecewise continuous functions (those which may have a jump across some surfaces or curves or points in B) are integrable.

Fubini's Theorem Let f be piecewise continuous in B. Then

$$\iiint_B f = \iint_R \int_e^f f(x,y,z)\,dz\,dA(x,y), \text{ where}$$

$R = [a,b] \times [c,d]$.

- A further reduction gives

$$\iiint_B f = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx.$$

- x, y, z can be exchanged. For instance,

$$\iiint_B f = \iint_{[c,d]\times[e,f]} \int_a^b f(x,y,z)\,dx\,dA(y,z).$$

- Triple Integrals over Regions.

A region in space consists of points bounded/enclosed by some surfaces.

We use Ω to denote a region in $\mathbb{R}^3$.

Just as in the 2-D case, define the "universal extension"

$$\tilde{f}(x,y,z) = \begin{cases} f(x,y,z), & (x,y,z) \in \Omega, \\ 0, & (x,y,z) \notin \Omega. \end{cases}$$

For f defined in Ω, define

$$\iiint_{\Omega} f \, dV = \iiint_{B} \tilde{f} \, dV.$$

Using this definition, when Ω can be expressed as

$$\{(x,y,z) : f_1(x,y) \le z \le f_2(x,y)\},$$
$$(x,y) \in D$$

Fubini's theorem takes the form

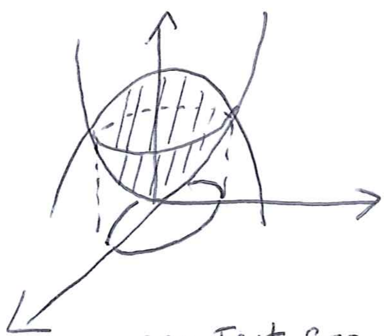
$$\iiint_{\Omega} f \, dV = \iint_{D} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z) \, dz \, dA(x,y). \quad (\star)$$

e.g. Find the volume of Ω enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

By sketching the surfaces one can see that Ω is bounded by f_1, f_2 like

$$x^2 + 3y^2 \le z \le 8 - x^2 - y^2$$

over some D.



see Text for a better picture.

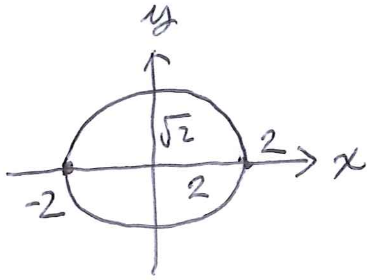
the boundary of D is those points $(x,y) \in \mathbb{R}^2$ so that (x,y,z) form the intersection of the two surfaces.

Solving

$$\begin{cases} z = x^2 + 3y^2 \\ z = 8 - x^2 - y^2 \end{cases}$$

we get $x^2 + 2y^2 = 4$, ie, D is

$$\{(x,y) : x^2 + 2y^2 \leq 4\}$$



$\therefore$ Vol of Ω

$$= \iint_D \int_{x^2+3y^2}^{8-x^2-y^2} 1 \, dz \, dA(x,y)$$

$$= \iint_D (8 - x^2 - y^2 - x^2 - 3y^2) \, dA(x,y)$$

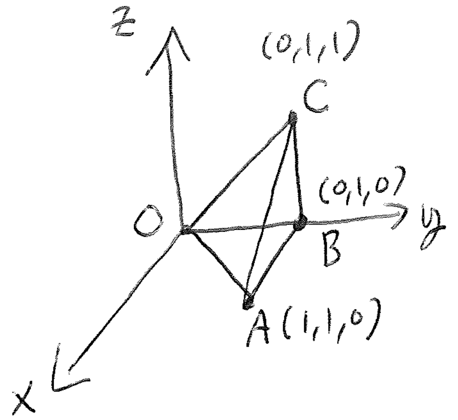
$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 - 4y^2) \, dy \, dx$$

$$= \int_{-2}^{2} \left(2(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{8}{3}\left(\frac{4-x^2}{2}\right)^{3/2} \right) dx$$

$$\vdots$$

$$= 8\pi\sqrt{2}.$$

e.g. Find the volume of the tetrahedron T with vertices $(0,0,0)$, $(1,1,0)$, $(0,1,0)$ and $(0,1,1)$.



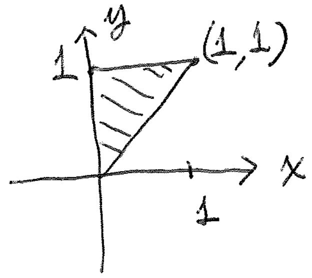
Four faces of T lie on

ABC on $y=1$
OAB on $z=0$ (xy-plane)
OBC on $x=0$ (yz-plane)
OAC on $x-y+z=0$

T can be expressed as

$$\left\{(x,y,z) : \begin{array}{l} 0 \le z \le y-x, \\ (x,y) \in D \end{array}\right\}$$

where D is



$\therefore$ volume of T :

$$\iiint_T 1\,dV = \iint_D \int_0^{y-x} 1\,dz\,dA(x,y)$$

$$= \iint_D (y-x)\,dA(x,y)$$

$$= \int_0^1 \int_x^1 (y-x)\,dy\,dx$$

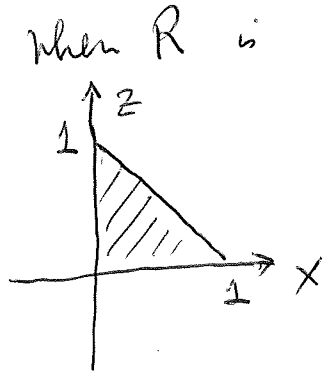
$$\vdots$$

$$= 1/6.$$

Or, T also expressed as

$$\left\{(x,y,z) : \begin{array}{l} x+z \le y \le 1, \\ (x,y) \in R \end{array}\right\}$$

when R is



$$\therefore \text{vol. of } T = \iint_R \int_{x+z}^{1} 1\,dy\,dA(x,z)$$

$$= \iint_R (1-x-z)\, dA(x,z)$$

$$= \int_0^1 \int_0^{1-x} (1-x-z)\, dz\, dx$$

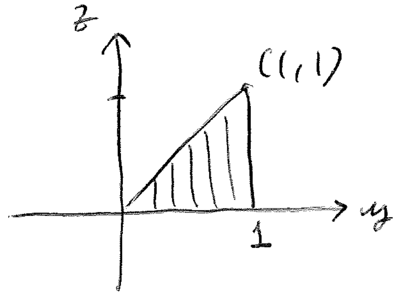
$$\vdots$$

$$= 1/6. \quad \text{the same result.}$$

One may also regard T as

$$\left\{ (x,y,z) : \begin{array}{l} 0 \le x \le y-z, \\ (y,z) \in R \end{array} \right\}$$

where R is



Volume of T

$$= \iint_R \int_0^{y-z} 1\, dx\, dA(y,z)$$

$$= \iint_R (y-z)\, dA(y,z)$$

$$= \int_0^1 \int_0^y (y-z)\, dz\, dy$$

$$= \int_0^1 \left(yz - \frac{z^2}{2} \right) \Big|_0^y dy$$

$$= \int_0^1 \frac{1}{2} y^2\, dy$$

$$= 1/6, \text{ again } \#$$

- A Digession

How to write down the quation of a plane through 3 given points.

Two cases : the plane passing through (0,0,0)
the plane not passing through (0,0,0).

In the first case, the equction is

$$ax + by + cz = 0$$

Use vector notation

$$\vec{n} \cdot \vec{x} = 0 \quad , \quad \vec{x} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}.$$

$P = \{(x,y,z) : \vec{n} \cdot \vec{x} = 0\}$, when $\vec{n}$ is a non-zero vectors.

When 2 points are given, $\vec{u}_1$ and $\vec{u}_2$, say. Then

$$\vec{n} = \vec{u}_1 \times \vec{u}_2.$$

(since $\vec{a} \times \vec{b}$ is always $\perp$ to $\vec{a}$ and $\vec{b}$.)

e.g. Find the equation of the plane passing (0,0,0), (1,1,2) and (0,0,-5).

Take $\vec{u}_1 = (1,1,2)$, $\vec{u}_2 = (0,0,-5)$

$$\vec{u}_1 \times \vec{u}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 2 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= -5\,i - (-5)\,j + 0\,k$$

$$= (-5, 5, 0).$$

$\therefore$ the equation is

$$-5x+5y=0$$

or $-x+y=0$.

Case 2. the equation is

$$ax+by+cz=d,\ d\neq 0.$$

or $\vec{n}\cdot\vec{x}=d$

When $\vec{u}_0, \vec{u}_1, \vec{u}_2$ are on the plane.

$\vec{v}_1=\vec{u}_1-\vec{u}_0,\ \vec{v}_2=\vec{u}_2-\vec{u}_0$ satisfy

$$\vec{n}\cdot\vec{v}_1=\vec{n}\cdot(\vec{u}_1-\vec{u}_0)=d-d=0$$

$$\vec{n}\cdot\vec{v}_2=0$$

So $\vec{n}=\vec{v}_1\times\vec{v}_2$.

e.g. Find the equation of the plane passing through $(1,0,0)$, $(0,2,-1)$, $(6,1,1)$.

Let $\vec{v}_1=(0,2,-1)-(1,0,0)=(-1,2,-1)$

$\vec{v}_2=(6,1,1)-(1,0,0)=(5,1,1)$

$$\vec{n}=\begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ 5 & 1 & 1 \end{vmatrix}=3i-4j+(-11)k$$

$$=(3,-4,-11)$$

$\therefore$ equation is $3x-4y-11z=d$

Since $(1,0,0)$ on the plane,

$$3\times1-4\times0-11\times0=d,\quad d=3$$

$\therefore$ it is $3x-4y-11z=3$.

- Some theoretical aspects: Pf of Fubini; volume\area; Another version of Fubini

Fubini's thm

$$\iiint_B f\,dV = \iint_R \int_e^f f(x,y,z)\,dz\,dA(x,y)$$

$$B = [a,b]\times[c,d]\times[e,f] = R\times[e,f]$$

Idea of Proof :

$$\iiint_B f\,dV \approx \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k \qquad (\star)$$

$$= \sum_{i,j} \left(\sum_k f(x_i^*, y_j^*, z_k^*)\,\Delta z_k\right) \Delta x_i \Delta y_j .$$

As $\|P\| \to 0$, $\displaystyle\sum_k f(x_i^*, y_j^*, z_k^*)\,\Delta z_k \longrightarrow \int_e^f f(x_i^*, y_j^*, z)\,dz$

Setting $F(x,y) = \displaystyle\int_e^f f(x,y,z)\,dz$, $\longrightarrow F(x_i^*, y_j^*)$

$$\therefore \iiint_B f\,dV \approx \sum_{i,j} F(x_i^*, y_j^*)\,\Delta x_i \Delta y_j \quad \text{as } \|P\|\to 0$$

$$\longrightarrow \iint_R F(x,y)\,dA(x,y)$$

$$= \iint_R \int_e^f f(x,y,z)\,dz\,dA(x,y).$$

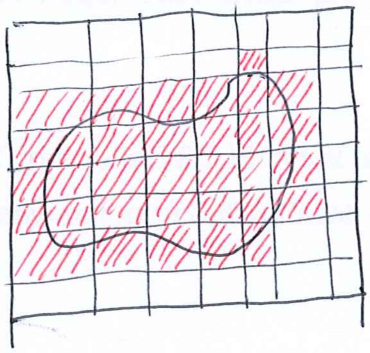
Next, we explain why for $D \subset \mathbb{R}^2$

$$|D| = \iint_D 1 \, dA$$

is the area of D, and, for $\Omega \subset \mathbb{R}^3$,

$$|\Omega| = \iiint_\Omega 1 \, dV$$

is the volume of Ω. It suffices to consider D since the case for Ω can be understood in the same way.



$\Omega \subset R$

Let P be a partition on R.

$\mathcal{A} = \{$ those R_{ij} contained completely inside $D \}$

$\mathcal{B} = \{$ those R_{ij}, $R_{ij} \cap D \neq \phi$, $R_{ij} \cap (R \setminus D) \neq \phi \}$

$\mathcal{C} = \{$ those R_{ij} contained in $R \setminus D \}$

$$R(\tilde{1}, P) = \sum_{\mathcal{A}} \tilde{1}(x_i^*, y_j^*) \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \tilde{1}(x_i^*, y_j^*) \cdot \Delta x_i \Delta y_j$$

where

$$\tilde{1} = \chi_D(x,y) = \begin{cases} 1, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$

$$R(\tilde{1}, P) = \sum_{\mathcal{A}} \Delta x_i \Delta y_j + \sum_{\mathcal{B}} \tilde{1}(x_i^*, y_j^*) \Delta x_i \Delta y_j$$

since $\tilde{1}(x,y) = 1$ on $\mathcal{A}$, $\tilde{1}(x,y) = 0$ on $\mathcal{C}$. For $R_{ij} \in \mathcal{B}$, When we take $(x_i^*, y_j^*) \in R_{ij} \cap D$, $\tilde{1}(x_i^*, y_j^*) = 1$, so

$$R(\tilde{1}, P) = \sum_{A, B} \Delta x_i \Delta y_j,$$

When we take $(x_i^*, y_j^*) \in R_{ij} \cap R \setminus D$, $\tilde{1}(x_i^*, y_j^*) = 0$, so

$$R(\tilde{1}, P') = \sum_{A} \Delta x_i \Delta y_j.$$

$R(\tilde{1}, P)$ is the area of the red part

$R(\tilde{1}, P')$ - - - - - - - - "green" part (imagine!)

Clearly,

green part $\leq$ area of $D \leq$ red part

As $\|P\| \to 0$, both red and green parts

$$\to \iint_R \tilde{1}\, dA = \iint_D 1\, dA. \quad \#$$

Third, we look at ☆ again

$$\iiint_B f\, dV \approx \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \Delta z_k$$

$$= \sum_k \left(\sum_{i,j} f(x_i^*, y_j^*, z_k^*) \Delta x_i \Delta y_j \right) \Delta z_k$$

(put bracket in a different way)

$$\to \int_e^f \iint_R f(x, y, z)\, dA(x, y)\, dz$$

When f is defined in a region Ω, we have

$$\iiint_{\Omega} f\,dV = \int_e^f \iint_{\Omega(z)} f(x,y,z)\,dA(x,y)\,dz,$$ where $\Omega(z)$ is the cross section of Ω at z, ie

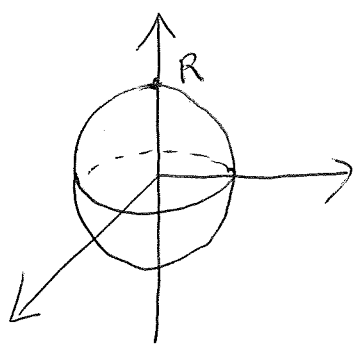
$$\Omega(z) = \{(x,y) : (x,y,z) \in \Omega\}.$$

When $f = 1$, we get the useful volume formula:

$$\begin{aligned}\text{vol of } \Omega &= |\Omega| \\ &= \iiint_{\Omega} 1\,dV \\ &= \int_e^f |\Omega(z)|\,dz,\end{aligned}$$ where $|\Omega(z)|$ is the area of the cross section $\Omega(z)$.

"Volume of Ω is the integral of the area of its cross sections."

e.g. Find the volume of the ball of radius R.



For $z \in [0, R]$, the cross section $B(z)$ is

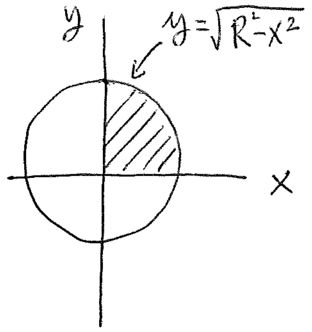
$$\begin{aligned}&\{(x,y) : (x,y,z) \in B\} \\ &= \{(x,y) : x^2+y^2+z^2 \le R^2\} \\ &= \{(x,y) : x^2+y^2 \le (R^2-z^2)\}\end{aligned}$$

ie, $B(z)$ is a disk of radius $\sqrt{R^2-z^2}$.

$$|B(z)| = \pi\left(\sqrt{R^2-z^2}\right)^2 = \pi(R^2-z^2).$$

$$\text{vol. of ball} = 2\int_0^R |B(z)|\,dz$$
$$= 2\int_0^R \pi(R^2-z^2)\,dz$$
$$= 2\pi\left(R^2 z - \frac{z^3}{3}\right)\Big|_0^R$$
$$= \frac{4\pi}{3}R^3. \quad \#$$

Using old approch we need to evaluate



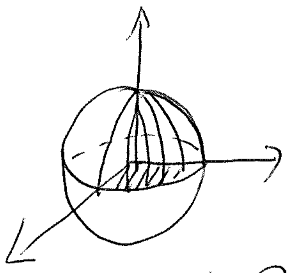
vol. of ball

$$= 8 \iint_D \int_0^{\sqrt{R^2-x^2-y^2}} dz\, dA(x,y) \quad , \text{ where}$$

D is the quater disk

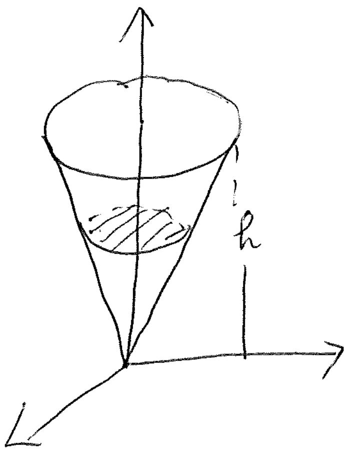
$$= 8\int_0^R \int_0^{\sqrt{R^2-x^2}} \int_0^{\sqrt{R^2-x^2-y^2}} dz\, dy\, dx$$

more complicated.



divide into 8 equal parts.

e.g. Find the volume of the circular cone $z=\sqrt{x^2+y^2}$ of height h.



The cross section $C(z)$ at $z\in[0,h]$

is $\{(x,y) : (x,y,z)\in C\}$

$= \{(x,y) : \sqrt{x^2+y^2}\le z\}$

$= \{(x,y) : x^2+y^2\le z^2\}$

$=$ disk of radius z

$\therefore$ volume of C

$$= \int_0^h |C(z)|\,dz$$

$$= \int_0^h \pi z^2\,dz$$

$$= \left.\frac{\pi z^3}{3}\right|_0^h$$

$$= \frac{1}{3}\pi h^3. \quad \#$$